Large time profile of solutions for a dissipative nonlinear evolution system with conservational form

2005 J. Phys. A: Math. Gen. 38 10955

(http://iopscience.iop.org/0305-4470/38/50/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 10955–10969 doi:10.1088/0305-4470/38/50/006

Large time profile of solutions for a dissipative nonlinear evolution system with conservational form

Zhian Wang

Department of Mathematical and Statistical Sciences, 632 CAB, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

E-mail: zhian@ualberta.ca

Received 16 May 2005, in final form 18 October 2005
Published 30 November 2005
Online at stacks.iop.org/JPhysA/38/10955

Abstract

We examine the Cauchy problem for a nonlinear dissipative evolution system with conservation form

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, & (t, x) \in [0, +\infty) \times R, \\ \theta_t = -(1 - \beta)\theta + \nu\psi_x + (\psi\theta)_x + \beta\theta_{xx}, \end{cases}$$

with initial condition

$$(\psi(0, x), \theta(0, x)) = (\psi_0(x), \theta_0(x)) \in H^1(R, R^2),$$

where α, β, σ and ν are positive constants such that $\alpha < \sigma$ and $\beta < 1$. We establish the global existence and decay rate of the solution subject to the parameter restriction $\nu < \frac{4\sqrt{\alpha\beta(1-\beta)(\sigma-\alpha)}}{\sigma}$. The optimal decay rate is obtained if $(\psi_0, \theta_0) \in L^1(R, R^2)$ furthermore. The global existence of the solution to the same problem has been studied in Jian and Chen (1998 *Acta Math. Sin.* **14** 17–34) without giving the decay rate and optimal decay order of the solution, in which $(\psi_0, \theta_0) \in H^1(R, R^2) \cap L^1(R, R^2)$.

PACS number: 02.30.−f

1. Introduction

In this paper, we consider the asymptotic behaviour of the Cauchy problem for a nonlinear dissipative evolution system with conservation form in one-dimensional spatial space

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, & (t, x) \in [0, +\infty) \times R, \\ \theta_t = -(1 - \beta)\theta + \nu\psi_x + (\psi\theta)_x + \beta\theta_{xx}, \end{cases} \tag{1.1}$$

subject to the initial data

$$(\psi(0,x),\theta(0,x)) = (\psi_0(x),\theta_0(x)). \tag{1.2}$$

System (1.1) was introduced by Hsieh [6] to observe the nonlinear interaction between ellipticity and dissipation. Indeed, by neglecting the damping and diffusion term, system (1.1) is reduced to

$$\begin{cases} \psi_t = -\sigma\theta_x, \\ \theta_t = \nu\psi_x. \end{cases} \tag{1.3}$$

It is clear that system (1.3) is elliptic and the equilibrium $(\psi,\theta) = 0$ is unstable. Taking the nonlinear term $(\psi\theta)_x$ into consideration, the system is still unstable if $|\psi| \ll 1$, and it will switch to be hyperbolic and the equilibrium will be stable whenever $|\psi| \gg 1$. Thus, a 'switching back and forth' phenomenon is expected due to the interplaying among ellipticity, hyperbolicity and dissipation. System (1.1) exhibits many applications to the realistic model. With the change of variables

$$\begin{cases} \psi = -\sqrt{2}X(t)\sin x, \\ \theta = \sqrt{2}Y(t)\cos x + 2Z(t)\cos 2x, \end{cases}$$

the Lorenz equation can be derived from (1.1) by only retaining the coefficients of $\sin x$, $\cos x$ and $\cos 2x$. Then the rich contents of the Lorenz equations are presumably also contained in system (1.1). By making $\alpha = \sigma = 1, \nu = 0, \beta = 1$ and replacing ψ by $-\psi$, system (1.1), then, is reduced to a chemotaxis model discussed in [10].

Tang and Zhao [16] studied the asymptotical solution for a slightly modified system by replacing the nonlinear term $(\psi\theta)_x$ in (1.1) with $2\psi\theta$. Jian and Chen [9] first established the global existence of solutions to system (1.1) when $(\psi_0,\theta_0) \in H^1(R,R^2)\cap L^1(R,R^2)$. Hsiao and Jian [7] obtained the global existence of classical solutions for the initial boundary value problem of system (1.1) with initial condition $(\psi_0,\theta_0) \in C^{2,\delta}([0,1]) * C^{2,\delta}([0,1]) (0 < \delta < 1)$ and periodic boundary condition

$$(\psi_0,\theta_0)(0) = (\psi_0,\theta_0)(1), \qquad ((\psi_0)_x,(\theta_0)_x)(0) = ((\psi_0)_x,(\theta_0)_x)(1), \quad 0 \leqslant t \leqslant T.$$

However, in [9], the restriction on the parameters α, β, σ and ν to ensure the global existence of the solution was not derived. They used only an abstract form instead of justifying the relationship between these parameters. It turns out from the linear analysis in section 3 that the global solution exists subject to the appropriate value of parameters chosen. One of the purposes of this paper is to give the relationship explicitly between the parameters which guarantees the global existence of the solution to system (1.1) with initial condition $(\psi_0,\theta_0) \in H^1(R,R^2)$. In addition, we deduce the decay rate of the solution and give the optimal decay order of the solution provided that $(\psi_0,\theta_0) \in L^1(R,R^2)$ furthermore.

We close the introduction by stating some notation.

Notation. Throughout the paper, if there is no ambiguity, we use C to denote generic positive constants which can change from line to line. When the dependence of the constant on some index or a function is important, we highlight it in the notation. $L^p(R)(1 \leqslant p \leqslant \infty)$ denotes usual Lebesgue space with the norm $\|f\|_{L^p(R)} = (\int_R |f(x)|^p\,\mathrm{d}x)^{\frac{1}{p}}, 1 \leqslant p < \infty$, as well as $\|f\|_{L^\infty(R)} = \sup_{x\in R}|f(x)|$. The space $L^p(R,R^2)$, for $1 \leqslant p \leqslant \infty$, denotes the usual Lebesgue space of R^2-valued functions equipped with norm $\|(u,v)\|_{L^p(R,R^2)} = \|u\|_{L^p(R)} + \|v\|_{L^p(R)}$. Moreover, we denote by $H^l(R)$ the usual lth-order Sobolev space with its norm $\|f\|_{H^l(R)} = \left(\sum_{i=0}^{l}\|\partial_x^i f\|^2\right)^{\frac{1}{2}}$. $\mathbb{Z}^+$ denotes the class of all positive integers and $\mathbb{N}$ denotes the set of all positive integers plus zero.

2. Global existence and decay estimates

In this section, we are going to prove the global existence and decay rate of solutions to system (1.1) using energy method with the initial condition $(\psi_0, \theta_0) \in L^2(R)$, which is based on a local existence and *a priori* estimates. First of all, we give the local existence as follows.

Theorem 2.1. *If $(\psi_0, \theta_0) \in H^1(R, R^2)$, then there exists a positive constant T depending only on the H^1-norm of initial data, such that the Cauchy problem (1.1), (1.2) admits a unique smooth solution $(\psi(t, x), \theta(t, x) \in H^1([0, T] \times R, R^2)$ satisfying*

$$\|(\psi(t, x), \theta(t, x)\|_{H^1([0,T]\times R,R^2)} \leqslant 2\|(\psi_0(x), \theta_0(x)\|_{H^1(R,R^2)}. \tag{2.1}$$

Proof. By the standard approach, we first rewrite system (1.1) in terms of an integral form as follows:

$$\begin{cases} \psi(t,x) = G^\alpha(t,x) * \psi_0(x) - (\sigma-\alpha)\int_0^t G^\alpha(t-s,x) * \psi(s,x)\,\mathrm{d}s \\ \qquad\qquad + \sigma\int_0^t G_x^\alpha(x, t-s) * \theta(s,x)\,\mathrm{d}s, \\ \theta(t,x) = G^\beta(t,x) * \theta_0(x) - (1-\beta)\int_0^t G^\beta(t-s,x) * \theta(s,x)\,\mathrm{d}s \\ \qquad\qquad - \nu\int_0^t G_x^\beta(x,t-s) * \psi(s,x)\,\mathrm{d}s - \int_0^t G_x^\beta(t-s,x) * (\psi\theta)(s,x)\,\mathrm{d}s, \end{cases} \tag{2.2}$$

where $G^\alpha(t,x) = \frac{1}{\sqrt{4\pi\alpha t}}\exp\left(-\frac{x^2}{4\alpha t}\right)$ and $G^\beta(t,x) = \frac{1}{\sqrt{4\pi\beta t}}\exp\left(-\frac{x^2}{4\beta t}\right)$ are fundamental solutions of heat equation and asterisk denotes the convolution which is taken with respect to the space variable x and subscript means the partial derivative.

Then the local existence can be obtained by using the contraction mapping principle to the integral representation (2.2), following the standard theory of parabolic equation. Although the operation will be somewhat complicated, the approach is routine and feasible. So we omit the details here. The details of the fundamental proof can be seen in [13]. □

To derive the *a priori* estimates, it is worthwhile to point out first that if $\nu < \frac{4\sqrt{\alpha\beta(1-\beta)(\sigma-\alpha)}}{\sigma}$, then we can find $\epsilon \in (0, 2)$, $C_0 > 0$ such that

$$\begin{cases} 2C_0\beta - \dfrac{\sigma^2}{\epsilon(\sigma-\alpha)} > 0, \\ 2(1-\beta) - \dfrac{C_0\nu^2}{\alpha\epsilon} > 0. \end{cases} \tag{2.3}$$

For the convenience of presentation, we give the following definition.

Definition 2.2. *Suppose that $\nu < \frac{4\sqrt{\alpha\beta(1-\beta)(\sigma-\alpha)}}{\sigma}$. A tupel of real parameters $(\alpha, \beta, \nu, \sigma, \epsilon, C_0)$ is called admissible if (2.3) holds for any $\alpha < \sigma$, $\beta < 1$, $C_0 > 0$ and $0 < \epsilon < 2$.*

Remark 2.3. It should be noted that the set of admissible parameters is not empty. Indeed, we can give an algorithm to find admissible parameters stepwise as follows.

Step 1. Define k with $0 < k < 1$ by $\nu\sigma = 4k\sqrt{\alpha\beta(1-\beta)(\sigma-\alpha)}$.

Step 2. Choose ϵ with $2k < \epsilon < 2$.

Step 3. Choose λ with $\lambda > 0$.

Step 4. Define C_0 by $C_0 = \frac{1}{1+\lambda}\left(\frac{\sigma^2}{2\beta(\sigma-\alpha)\epsilon} + \lambda\frac{2\alpha(1-\beta)\epsilon}{\nu^2}\right)$.

Before embarking on the *a priori* estimates, we would like to give the following inequality which will be applied later.

Lemma 2.4 (Ehrling–Browder's inequality [1]). *Let $v \in L^q(R^N, R^n)$ and $D^m v \in L^r(R^N, R^n)$ with $1 \leqslant q, r \leqslant \infty$. Then for any integer j with $0 \leqslant j \leqslant m$, there exists a constant $C = C(m, p, N)$, such that*

$$\|D^j v\|_{L^p(R^N,R^n)} \leqslant C\|D^m v\|^{\alpha}_{L^r(R^N,R^n)}\|v\|^{1-\alpha}_{L^q(R^N,R^n)}, \tag{2.4}$$

where p is determined by

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-\alpha}{q}, \qquad \frac{j}{m} \leqslant \alpha \leqslant 1.$$

Lemma 2.5. *Assume that $v \in W^{1,2}(R)$, i.e., $v \in L^2(R)$ and $v_x \in L^2(R)$; then $v \in L^4(R)$ and satisfies*

$$\|v\|^4_{L^4(R)} \leqslant \|v_x\|_{L^2(R)}\|v\|^3_{L^2(R)}. \tag{2.5}$$

Proof. First noticing the fact that $W^{1,2}(R) = W^{1,2}_0(R)$ (see [1]), which tells us that $v(-\infty) = v(+\infty) = 0$, then it follows that

$$v^2(x) = \int_{-\infty}^{x} (v^2)_x\,\mathrm{d}x \leqslant 2\int_{-\infty}^{x} |vv_x|\,\mathrm{d}x,$$

and

$$v^2(x) = -\int_{x}^{+\infty} (v^2)_x\,\mathrm{d}x \leqslant 2\int_{x}^{+\infty} |vv_x|\,\mathrm{d}x.$$

The combination of these two inequalities gives us that

$$v^2(x) \leqslant \int_{-\infty}^{+\infty} |vv_x|\,\mathrm{d}x = \int_R |vv_x|\,\mathrm{d}x.$$

Then Cauchy–Schwarz inequality yields that

$$\begin{aligned}\int_R v^4\,\mathrm{d}x &\leqslant \int_R v^2\left(\int_R |vv_x|\,\mathrm{d}x\right)\mathrm{d}x \leqslant \int_R v^2\,\mathrm{d}x \int_R |vv_x|\,\mathrm{d}x \\ &\leqslant \int_R v^2\,\mathrm{d}x \left(\int_R v^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_R v_x^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \left(\int_R v_x^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_R v^2\,\mathrm{d}x\right)^{\frac{3}{2}},\end{aligned}$$

which completes the proof. □

Remark 2.6. Indeed, inequality (2.5) can be regarded as a corollary of Ehrling–Browder's inequality by taking $j = 0$, $N = 1$, $p = 4$, $m = 1$, $\alpha = \frac{1}{4}$, $r = 2$ and $q = 2$ in (2.4). However, it is crucial in our analysis to figure out the constant C in inequality (2.4). Therefore, we exhibit the proof here to identify this constant with 1 in lemma 2.4.

Now we are in a position to give the *a priori* estimates as follows.

Lemma 2.7. *Let $\alpha, \beta, \nu, \sigma, \epsilon$ and C_0 belong to the set of admissible parameters such that $\nu < \frac{4\sqrt{\alpha\beta(1-\beta)(\sigma-\alpha)}}{\sigma}$. Assume that $(\psi(t,x)\theta(t,x))$ is a solution to (1.1), (1.2) obtained in theorem 2.1 with initial data $(\psi_0, \theta_0) \in H^1(R, R^2)$, satisfying*

$$\left[\int_R \left(\psi_0^2(x) + C_0\theta^2(x)\right)\mathrm{d}x\right]^2 \leqslant \frac{4l\alpha^2\epsilon'^2\epsilon''}{C_0} \tag{2.6}$$

for any $\epsilon' \in (0, 2-\epsilon)$ *and* $\epsilon'' \in \left(0, 2\beta C_0 - \frac{\sigma^2}{\epsilon(\sigma-\alpha)}\right)$*; then there exists a constant* C *dependent on initial data and the parameters* α, β, σ *and* ν *such that*

$$\|(\psi(t,x), \theta(t,x))\|^2_{H^1(R,R^2)} \leqslant C\,\mathrm{e}^{-lt}, \tag{2.7}$$

where

$$l = \min\left\{(2-\epsilon)(\sigma-\alpha), 2\left(1-\beta-\frac{\nu^2}{2\epsilon\alpha}\right)\right\}. \tag{2.8}$$

Proof. Multiplying the first equation of (1.1) by 2ψ and the second equation of (1.1) by $2C_0\theta$ and integrating the resulting identity with respect to x over R, we get by the use of integration by parts with Cauchy–Schwarz inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_R(\psi^2+C_0\theta^2)\,\mathrm{d}x + 2(\sigma-\alpha)\int_R\psi^2\,\mathrm{d}x + 2(1-\beta)C_0\int_R\theta^2\,\mathrm{d}x\\
&\qquad +2\alpha\int_R\psi_x^2\,\mathrm{d}x + 2\beta C_0\int_R\theta_x^2\,\mathrm{d}x\\
&\quad = -2\sigma\int_R\psi\theta_x\,\mathrm{d}x + 2\nu C_0\int_R\theta\psi_x\,\mathrm{d}x + 2C_0\int_R\theta(\psi\theta)_x\,\mathrm{d}x\\
&\quad = -2\sigma\int_R\psi\theta_x\,\mathrm{d}x + 2\nu C_0\int_R\theta\psi_x\,\mathrm{d}x + C_0\int_R\psi_x\theta^2\,\mathrm{d}x\\
&\quad \leqslant \epsilon(\sigma-\alpha)\int_R\psi^2\,\mathrm{d}x + \frac{\sigma^2}{\epsilon(\sigma-\alpha)}\int_R\theta_x^2\,\mathrm{d}x + \epsilon\alpha\int_R\psi_x^2\,\mathrm{d}x + \frac{\nu^2C_0^2}{\epsilon\alpha}\int_R\theta^2\,\mathrm{d}x\\
&\qquad +\alpha\epsilon'\int_R\psi_x^2 + \frac{C_0^2}{\alpha\epsilon'}\int_R\theta^4\,\mathrm{d}x,
\end{aligned}\tag{2.9}$$

where ϵ' is a constant valued between 0 and $2-\epsilon$.

We rearrange the terms in (2.9) to obtain from inequality (2.5) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_R(\psi^2+C_0\theta^2)\,\mathrm{d}x + (2-\epsilon)(\sigma-\alpha)\int_R\psi^2\,\mathrm{d}x + 2\left(1-\beta-\frac{\nu^2C_0}{\epsilon\alpha}\right)C_0\int_R\theta^2\,\mathrm{d}x\\
&\qquad +(2-\epsilon-\epsilon')\alpha\int_R\psi_x^2\,\mathrm{d}x + \left(2\beta C_0 - \frac{\sigma^2}{\epsilon(\sigma-\alpha)}\right)\int_R\theta_x^2\,\mathrm{d}x\\
&\quad \leqslant \frac{C_0^2}{\alpha\epsilon'}\int_R\theta^4\,\mathrm{d}x \leqslant \frac{C_0^2}{\alpha\epsilon'}\|\theta_x\|_{L^2(R)}\|\theta\|^3_{L^2(R)}\\
&\quad \leqslant \epsilon''\int_R\theta_x^2\,\mathrm{d}x + \frac{C_0^4}{4\alpha^2\epsilon'^2\epsilon''}\left(\int_R\theta(t,x)^2\,\mathrm{d}x\right)^3.
\end{aligned}\tag{2.10}$$

Here we have applied the Young inequality in the last inequality and ϵ'' is a constant such that $\epsilon'' \in \left(0, 2\beta C_0 - \frac{\sigma^2}{\epsilon(\sigma-\alpha)}\right)$.

Defining $L = 2\beta C_0 - \frac{\sigma^2}{\epsilon(\sigma-\alpha)} - \epsilon'' > 0$ and shuffling the terms in (2.10), together with the definition of l, we end up with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_R(\psi^2(t,x)+C_0\theta^2(t,x))\,\mathrm{d}x + l\int_R(\psi^2(t,x)+C_0\theta^2(t,x))\,\mathrm{d}x\\
&\quad + L\int_R\left(\psi_x^2(t,x)+C_0\theta_x^2(t,x)\right)\mathrm{d}x \leqslant \frac{C_0^4}{4l\alpha^2\epsilon'^2\epsilon''}\left(\int_R\theta(t,x)^2\,\mathrm{d}x\right)^3.
\end{aligned}\tag{2.11}$$

To derive (2.7), we define $W(t) = \int_R \mathrm{e}^{lt}(\psi^2(t,x)+C_0\theta^2(t,x))\,\mathrm{d}x$, and take the derivative with respect to t. Then we obtain the following inequality from (2.11):

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) \leqslant \frac{C_0\,\mathrm{e}^{-2lt}}{4\alpha^2\epsilon'^2\epsilon''}W^3(t).$$

That is,

$$\frac{\mathrm{d}W(t)}{W^3(t)} \leqslant \frac{C_0}{4\alpha^2\epsilon'^2\epsilon''}\,\mathrm{e}^{-2lt}\,\mathrm{d}t. \tag{2.12}$$

Integration of (2.12) with respect to t over $[0, t]$ yields

$$\frac{1}{W^2(t)} \geqslant \frac{1}{W^2(0)} - \frac{C_0}{4l\alpha^2\epsilon'^2\epsilon''}. \tag{2.13}$$

Noting that $W(0) = \int_R \left(\psi_0^2(x) + \theta_0^2(x)\right)\mathrm{d}x$ and the assumption on initial data (2.6), we know from (2.13) that $W(t)$ is bounded. That is, there exists a constant C depending on $\alpha, C_0, l, \epsilon', \epsilon''$ and initial data, such that $W(t) = \mathrm{e}^{lt}\int_R(\psi^2(t,x) + C_0\theta^2(t,x))\,\mathrm{d}x < C$, which implies that

$$\int_R (\psi^2(t,x) + C_0\theta^2(t,x))\,\mathrm{d}x \leqslant C\,\mathrm{e}^{-lt}. \tag{2.14}$$

We therefore have $\left(\int_R \theta(t,x)^2\,\mathrm{d}x\right)^3 \leqslant C\,\mathrm{e}^{-3lt}$. Substituting this into (2.10) and integrating the resulting inequality with respect to t, we obtain the following estimates:

$$\int_0^t \int_R \left(\psi^2(t,x) + \psi_x^2(t,x) + \theta^2(t,x) + \theta_x^2(t,x)\right)\mathrm{d}x\,\mathrm{d}t \leqslant C(1 + \|(\psi_0(x), \theta_0(x))\|_{L^2(R,R^2)}). \tag{2.15}$$

To derive the desired estimate (2.7), we need to deduce the decay rate on the first-order derivative of solutions. Towards this end, we multiply the first equation of (1.1) by $(-2\psi_{xx})$ and the second equation by $(-2\theta_{xx})$, and add the resulting equations together. After all these, we take integration with respect to x over R and obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_R \left(\psi_x^2 + \theta_x^2\right)\mathrm{d}x + 2(\sigma-\alpha)\int_R \psi_x^2\,\mathrm{d}x + 2(1-\beta)\int_R \theta_x^2\,\mathrm{d}x + 2\alpha\int_R \psi_{xx}^2\,\mathrm{d}x + 2\beta\int_R \theta_{xx}^2\,\mathrm{d}x \\
&\quad = 2\sigma\int_R \psi_{xx}\theta_x\,\mathrm{d}x - 2\nu\int_R \psi_x\theta_{xx}\,\mathrm{d}x - 2\int_R \theta_{xx}^2(\psi\theta)_x\,\mathrm{d}x \\
&\quad \leqslant \alpha\int_R \psi_{xx}^2\,\mathrm{d}x + \frac{\sigma^2}{\alpha}\int_R \theta_x^2\,\mathrm{d}x + \frac{\beta}{2}\int_R \theta_{xx}^2\,\mathrm{d}x + \frac{2\nu^2}{\beta}\int_R \psi_x^2\,\mathrm{d}x \\
&\qquad + \frac{\beta}{2}\int_R \theta_{xx}^2\,\mathrm{d}x + \frac{2}{\beta}\int_R (\psi_x\theta + \psi\theta_x)^2\,\mathrm{d}x \\
&\quad \leqslant \alpha\int_R \psi_{xx}^2\,\mathrm{d}x + \frac{\sigma^2}{\alpha}\int_R \theta_x^2\,\mathrm{d}x + \beta\int_R \theta_{xx}^2\,\mathrm{d}x + \frac{2\nu^2}{\beta}\int_R \psi_x^2\,\mathrm{d}x \\
&\qquad + \frac{4}{\beta}\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_R \theta_x^2\,\mathrm{d}x + \frac{4}{\beta}\|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_R \psi_x^2\,\mathrm{d}x.
\end{aligned} \tag{2.16}$$

We shuffle the terms in (2.16) and integrate the resulting inequality with respect to t over $[0, t]$ to get

$$\begin{aligned}
&\int_R \left(\psi_x^2 + \theta_x^2\right)\mathrm{d}x + 2(\sigma-\alpha)\int_0^t\int_R \psi_x^2\,\mathrm{d}x\,\mathrm{d}t + 2(1-\beta)\int_0^t\int_R \theta_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \alpha\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \|\psi_x(0,x), \theta_x(0,x)\|_{L^2(R,R^2)} + \frac{\sigma^2}{\alpha}\int_0^t\int_R \theta_x^2\,\mathrm{d}x\,\mathrm{d}t + \frac{2\nu^2}{\beta}\int_0^t\int_R \psi_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{4}{\beta}\left(\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right)\int_0^t\int_R \left(\psi_x^2 + \theta_x^2\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.17}$$

The combination of (2.17) with (2.15) results in that

$$\int_R \left(\psi_x^2+\theta_x^2\right) \mathrm{d}x + 2(\sigma-\alpha)\int_0^t\int_R \psi_x^2 \,\mathrm{d}x\,\mathrm{d}t + 2(1-\beta)\int_0^t\int_R \theta_x^2\,\mathrm{d}x\,\mathrm{d}t \\ +\alpha\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant C\left[1+\|\psi(t,x)\|^2_{L^\infty([0,\infty]\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty]\times R)}\right]. \tag{2.18}$$

To obtain the desired estimates, we need to prove that the terms on the right-hand side of (2.18) are bounded. Towards this end, we make the term on the left-hand side smaller and arrive at

$$\int_R \left(\psi_x^2(t,x)+\theta_x^2(t,x)\right)\mathrm{d}x \leqslant C\left[1+\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right],$$

where the constant C depends on the parameters α, β, σ and ν. Applying the Hölder inequality, and taking (2.14) into consideration, one deduces that from the above inequality

$$\psi^2(t,x)+\theta^2(t,x) = \int_{-\infty}^x (\psi^2(t,\xi)+\theta^2(t,\xi))_\xi\,\mathrm{d}\xi \\ \leqslant 2\|\psi(t,x)\|_{L^2(R)}\|\psi_x(t,x)\|_{L^2(R)} + 2\|\theta(t,x)\|_{L^2(R)}\|\theta_x(t,x)\|_{L^2(R)} \\ \leqslant C\left[1+\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right]^{\frac{1}{2}}.$$

This implies that

$$\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)} \\ \leqslant C\left[1+\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right]^{\frac{1}{2}}.$$

Solving this inequality gives

$$\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)} \leqslant C. \tag{2.19}$$

Applying (2.19) in (2.18) yields that

$$\int_R \left(\psi_x^2+\theta_x^2\right) \mathrm{d}x + 2(\sigma-\alpha)\int_0^t\int_R \psi_x^2 \,\mathrm{d}x\,\mathrm{d}t + 2(1-\beta)\int_0^t\int_R \theta_x^2\,\mathrm{d}x\,\mathrm{d}t \\ +\alpha\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.20}$$

With definition l of (2.8), it is straightforward to deduce that

$$\int_R \left(\psi_x^2+\theta_x^2\right) \mathrm{d}x + l\int_0^t\int_R \left(\psi_x^2+\theta_x^2\right)\mathrm{d}x\,\mathrm{d}t + \alpha\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.21}$$

Utilizing Gronwall's inequality in (2.21) we get

$$\int_R \left(\psi_x^2(t,x)+\theta_x^2(t,x)\right)\mathrm{d}x \leqslant C\,\mathrm{e}^{-lt}. \tag{2.22}$$

Thus the combination of (2.14) and (2.22) gives (2.7) which completes the proof. □

Based on the local existence theorem and the *a priori* estimates, we obtain the following global existence theorem with decay estimates.

Theorem 2.8. *Let the assumptions hold in theorem 2.7. Then there exists a unique global-in-time solution* $(\psi,\theta) \in H^1([0,\infty)\times R, R^2)$ *to system (1.1) such that the solution decays exponentially in* H^1*-norm in the form*

$$\|(\psi(t,x),\theta(t,x)\|^2_{H^1(R,R^2)} \leqslant C\,\mathrm{e}^{-lt}. \tag{2.23}$$

An easy but useful observation is that the decay order of the solution in (2.23) will hold for any order derivative of the solution if the corresponding higher order derivative of initial data is in $H^1(R, R^2)$. To justify this, we only need to show that the exponential decay order holds for second-order derivative of the solution and higher order can be derived likewise. Indeed, we differentiate system (1.1) twice with respect to x, multiply them with $2\psi_{xx}$ and $2\theta_{xx}$, respectively, and add the resulting equation together. After these procedures, we take integration with respect to x over R and reach

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_R \left(\psi_{xx}^2+\theta_{xx}^2\right)\mathrm{d}x &+ 2(\sigma-\alpha)\int_R \psi_{xx}^2\,\mathrm{d}x + 2(1-\beta)\int_R \theta_{xx}^2\,\mathrm{d}x \\
&+2\alpha\int_R \psi_{xxx}^2\,\mathrm{d}x + 2\beta\int_R \theta_{xxx}^2\,\mathrm{d}x \\
&= -2\sigma\int_R \psi_{xx}\theta_{xxx}\,\mathrm{d}x + 2\nu\int_R \psi_{xxx}\theta_{xx}\,\mathrm{d}x + 2\int_R \theta_{xx}(\psi\theta)_{xxx}\,\mathrm{d}x \\
&= -2\sigma\int_R \psi_{xx}\theta_{xxx}\,\mathrm{d}x + 2\nu\int_R \psi_{xxx}\theta_{xx}\,\mathrm{d}x - 2\int_R \theta_{xxx}(\psi\theta)_{xx}\,\mathrm{d}x \\
&\leqslant \frac{\beta}{2}\int_R \theta_{xxx}^2\,\mathrm{d}x + \frac{2\sigma^2}{\beta}\int_R \psi_{xx}^2\,\mathrm{d}x + \alpha\int_R \psi_{xxx}^2\,\mathrm{d}x + \frac{\nu^2}{\alpha}\int_R \theta_{xx}^2\,\mathrm{d}x \\
&+\frac{\beta}{2}\int_R \theta_{xxx}^2\,\mathrm{d}x + \frac{2}{\beta}\int_R [(\psi\theta)_{xx}]^2\,\mathrm{d}x.
\end{aligned} \tag{2.24}$$

Note that it is easy to derive by the Cauchy–Schwarz inequality that

$$\begin{aligned}
\int_R [(\psi\theta)_{xx}]^2\,\mathrm{d}x &\leqslant C\int_R [(\psi_{xx}\theta)^2 + (2\psi_x\theta_x)^2 + (\psi\theta_{xx})^2]\,\mathrm{d}x \\
&\leqslant C\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_R \theta_{xx}^2\,\mathrm{d}x + C\|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_R \psi_{xx}^2\,\mathrm{d}x \\
&+C\|\theta_x(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_R \psi_x^2\,\mathrm{d}x.
\end{aligned} \tag{2.25}$$

The substitution of (2.25) into (2.24) and integration with respect to t over $[0, t]$ give

$$\begin{aligned}
\int_0^t\int_R \left(\psi_{xx}^2+\theta_{xx}^2\right)\mathrm{d}x\,\mathrm{d}t &+ 2(\sigma-\alpha)\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + 2(1-\beta)\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \\
&+\alpha\int_0^t\int_R \psi_{xxx}^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_0^t\int_R \theta_{xxx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_1 + \frac{2\sigma^2}{\beta}\int_0^t\int_R \psi_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \\
&+\frac{\nu^2}{\alpha}\int_0^t\int_R \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + C\|\theta_x(t,x)\|^2_{L^\infty([0,\infty)\times R)}\int_0^t\int_R \psi_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&+C\left\{\|\psi(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right\}\int_0^t\int_R \left(\psi_{xx}^2+\theta_{xx}^2\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.26}$$

where the constant C_1 is dependent of initial data of the second-order derivative.

Taking (2.15) and (2.20) into consideration, and remembering the definition of l, we end up with the following inequality:

$$\begin{aligned}
\int_R \left(\psi_{xx}^2(t,x)+\theta_{xx}^2(t,x)\right)\mathrm{d}x &+ l\int_0^t\int_R \left(\psi_{xx}^2(t,x)+\theta_{xx}^2(t,x)\right)\mathrm{d}x\,\mathrm{d}t \\
&+(\alpha+\beta)\int_0^t\int_R \left(\psi_{xxx}^2(t,x)+\theta_{xxx}^2(t,x)\right)\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C\left[1+\|\psi(t,x)\|^2_{L^\infty([0,\infty),R)} + \|\theta(t,x)\|^2_{L^\infty([0,\infty)\times R)}\right].
\end{aligned} \tag{2.27}$$

Here the constant C depends on the parameters α, β, σ, ν and initial data.

Performing the same argument as in the proof of lemma 2.7, it is easy to derive that

$$\|\psi_x(t,x)\|^2_{L^\infty([0,\infty)\times R)} + \|\theta_x(t,x)\|^2_{L^\infty([0,\infty)\times R)} \leqslant C. \tag{2.28}$$

Applying (2.28) in (2.27) gives

$$\int_R \left(\psi_{xx}^2(t,x) + \theta_{xx}^2(t,x)\right) \mathrm{d}x + l\int_0^t\int_R \left(\psi_{xx}^2(t,x) + \theta_{xx}^2(t,x)\right) \mathrm{d}x\,\mathrm{d}t$$
$$+ (\alpha+\beta)\int_0^t\int_R \left(\psi_{xxx}^2(t,x) + \theta_{xxx}^2(t,x)\right) \mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.29}$$

Again, the application of Gronwall's inequality in (2.29) leads to the decay rate for the second-order estimate

$$\int_R \left(\psi_{xx}^2(t,x) + \theta_{xx}^2(t,x)\right) \mathrm{d}x \leqslant C\,\mathrm{e}^{-lt}.$$

Taking the same procedure to higher order term, we obtain the following lemma.

Lemma 2.9. *Let (ψ, θ) be the solution to system (1.1). Then for $k = 0, 1, 2, \ldots,$ it follows that*

$$\left\| \frac{\partial^k}{\partial x^k}(\psi(t,x), \theta(t,x)) \right\|^2_{L^2(R,R^2)} \leqslant C\,\mathrm{e}^{-lt}. \tag{2.30}$$

3. Linear analysis

In this and next section, we are devoted to examining the optimal decay estimates of the solution to the special case of system (1.1) where $\alpha = \beta$ and $\sigma = 1$, which reads

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + (\psi\theta)_x + \alpha\theta_{xx}, \end{cases} \tag{3.1}$$

with initial data

$$(\psi(0,x), \theta(0,x)) = (\psi_0(x), \theta_0(x)), \tag{3.2}$$

where α and ν are positive constants such that $\alpha < 1$, $\nu < 4\alpha(1-\alpha)$.

In order to examine the optimal decay estimates of the solution to system (3.1), we need to first investigate the decay order of the solution to its corresponding linearized system which takes the form

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + \alpha\theta_{xx}. \end{cases} \tag{3.3}$$

Performing the Fourier transform to (3.3), we convert (3.3) into the following ordinary differential equations (ODEs):

$$\begin{pmatrix} \hat{\psi} \\ \hat{\theta} \end{pmatrix}_t = -\begin{pmatrix} 1-\alpha+\alpha\xi^2 & \mathrm{i}\xi \\ -\mathrm{i}\nu\xi & 1-\alpha+\alpha\xi^2 \end{pmatrix} \begin{pmatrix} \hat{\psi} \\ \hat{\theta} \end{pmatrix}, \tag{3.4}$$

with initial data

$$(\hat{\psi}(0,x), \hat{\theta}(0,x)) = (\hat{\psi}_0(x), \hat{\theta}_0(x)), \tag{3.5}$$

where $\hat{\psi}$ and $\hat{\theta}$ denote the Fourier transform with the frequency ξ. It is easy to calculate the eigenvalue λ of coefficient matrix taking values $\lambda = -(1-\alpha+\alpha\xi^2) \pm \sqrt{\nu}\xi$. Consequently, the solution of (3.4) and (3.5) is given by

$$\begin{cases} \hat{\psi} = \dfrac{e^{-(1-\alpha+\alpha\xi^2)t}}{2i\sqrt{\nu}}[e^{\sqrt{\nu}\xi t}(\hat{\theta}_0 + i\sqrt{\nu}\hat{\psi}_0) - e^{-\sqrt{\nu}\xi t}(\hat{\theta}_0 - i\sqrt{\nu}\hat{\psi}_0)], \\ \hat{\theta} = \dfrac{e^{-(1-\alpha+\alpha\xi^2)t}}{2}[e^{-\sqrt{\nu}\xi t}(\hat{\theta}_0 - i\sqrt{\nu}\hat{\psi}_0) + e^{\sqrt{\nu}\xi t}(\hat{\theta}_0 + i\sqrt{\nu}\hat{\psi}_0)]. \end{cases}$$

Thus, the inverse of Fourier transform gives the solution of (3.4) and (3.5) explicitly as follows:

$$\begin{cases} \psi(t,x) = \dfrac{1}{2i\sqrt{\nu}}[K_-(t,x) * (\theta_0 + i\sqrt{\nu}\psi_0) - K_+(t,x) * (\theta_0 - i\sqrt{\nu}\psi_0)], \\ \theta(t,x) = \dfrac{1}{2}[K_+(t,x) * (\theta_0 - i\sqrt{\nu}\psi_0) + K_-(t,x) * (\theta_0 + i\sqrt{\nu}\psi_0)], \end{cases} \tag{3.6}$$

where the kernel function $K_\pm(t,x)$ is given by

$$\begin{aligned} K_\pm(t,x) &= \mathcal{F}^{-1}\left[e^{-(1-\alpha+\alpha\xi^2\pm\sqrt{\nu}\xi)t}\right] = e^{-(1-\alpha)t}\mathcal{F}^{-1}\left[e^{-\alpha\xi^2 t\mp\sqrt{\nu}\xi t}\right] \\ &= \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\left(1-\alpha-\frac{\nu}{4\alpha}\right)t\right) \exp\left(\mp\frac{i\sqrt{\nu}}{2\alpha}x\right) \exp\left(-\frac{x^2}{4\alpha t}\right). \end{aligned} \tag{3.7}$$

Here $\mathcal{F}^{-1}$ means the inverse Fourier transform.

From the explicit expression of the solution (3.6) and kernel function $K_\pm(t,x)$ in (3.7), we observe that system (3.1) is stable if and only if $\nu < 4\alpha(1-\alpha)$. Moreover, when $p \in [1, +\infty]$ and $t \to +\infty$, it follows that

$$\begin{aligned} \|K_\pm(t,x)\|_{L^p(R)} &= \left\| \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\left(1-\alpha-\frac{\nu}{4\alpha}\right)t\right) \exp\left(-\frac{x^2}{4\alpha t}\right) \right\|_{L^p(R)} \\ &= O(1)t^{-\frac{1}{2}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t} \left\| \exp\left(-\frac{x^2}{4\alpha t}\right) \right\|_{L^p(R)} \\ &= O(1)t^{-\frac{1}{2}+\frac{1}{2p}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t}. \end{aligned}$$

Hence, it is straightforward to derive that

$$\begin{aligned} \left\| \frac{\partial}{\partial x} K_\pm(t,x) \right\|_{L^p(R)} &= \left\| \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\left(1-\alpha-\frac{\nu}{4\alpha}\right)t\right) \right. \\ &\quad \left. \times \left(\mp\frac{i\sqrt{\nu}}{2\alpha} \exp\left\{-\frac{x^2}{4\alpha t}\right\} + \frac{\partial}{\partial x} \exp\left\{-\frac{x^2}{4\alpha t}\right\}\right) \right\|_{L^p(R)} \\ &\leqslant Ct^{-\frac{1}{2}+\frac{1}{2p}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t} + Ct^{-1+\frac{1}{2p}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t} \\ &\leqslant Ct^{-\frac{1}{2}+\frac{1}{2p}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t}. \end{aligned}$$

Actually, using mathematical induction, we may easily prove that the above optimal decay rate holds for $(\partial^k/\partial x^k)K_\pm(t,x)$ for any $k \in \mathbb{N}$ and $p \in [1,\infty]$. That is,

$$\left\| \frac{\partial^k}{\partial x^k} K_\pm(t,x) \right\|_{L^p(R)} \leqslant Ct^{-\frac{1}{2}+\frac{1}{2p}} e^{-(1-\alpha-\frac{\nu}{4\alpha})t}. \tag{3.8}$$

From the solution expression (3.6), we find that the decay rate of the solution of (3.1), (3.2) is indicated by the decay rates of $K_\pm(t,x)$. Namely, (3.8) holds for the solution $(\psi(t,x), \theta(t,x))$ of (3.1) as well. We summarize in the following theorem.

Theorem 3.1. *Suppose that* $(\psi_0, \theta_0) \in L^1(R, R^2) \cap L^p(R, R^2)$*; then there exists a unique smooth solution* $(\psi(t, x), \theta(t, x))$ *to system (3.1), (3.2), which decays if and only if* $\nu < 4\alpha(1-\alpha)$*. Moreover, for any* $k \in \mathbb{N}$ *and* $p \in [1, \infty]$*, the following decay rate is optimal:*

$$\left\| \frac{\partial^k}{\partial x^k}(u(t,x), v(t,x)) \right\|_{L^p(R,R^2)} \leqslant C(1+t)^{-\frac{1}{2}+\frac{1}{2p}} \,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}.$$

4. Optimal decay estimates

In this section, we are devoted to proving that the solution of nonlinear system (3.1) has the same decay rate as that of the corresponding linearized system (3.2). That is, the solutions of system (3.1) decay with the optimal decay order. The main result is given in the following theorem.

Theorem 4.1. *Assume that* $\nu < 4\alpha(1-\alpha)$ *and* $(\psi_0, \theta_0) \in H^1(R, R^2) \cap L^1(R, R^2)$*. Then for any* $k \in \mathbb{N}$ *and p with* $1 \leqslant p \leqslant \infty$*, the solutions of (3.1), (3.2) decay with the following optimal decay order:*

$$\left\| \frac{\partial^k}{\partial x^k}(\psi(t,\cdot), \theta(t,\cdot)) \right\|_{L^p(R,R^2)} \leqslant C(1+t)^{-\frac{1}{2}+\frac{1}{2p}} \,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}, \tag{4.1}$$

where the constant C depends on α, ν, k and initial data.

To prove theorem 4.1, we need the following lemmas which will be essentially applied in this section.

Lemma 4.2. *For any* $k \in \mathbb{N}$*, there exists a constant C, such that the solution of (3.1), (3.2) satisfies*

$$\left\| \frac{\partial^k}{\partial x^k}(\psi(t,x), \theta(t,x)) \right\|_{L^\infty(R,R^2)} \leqslant C\,\mathrm{e}^{-lt/2}. \tag{4.2}$$

Proof. By the Sobolev inequality $\|u\|_{L^\infty(R)} \leqslant \|u\|_{L^2(R)}^{\frac{1}{2}} \|u_x\|_{L^2(R)}^{\frac{1}{2}}$, inequality (4.1) follows directly from inequality (2.30). □

Lemma 4.3. *Let* γ *and* η *be positive numbers,* $t > 0$*. Then it holds that*

$$\int_0^t (1+t-s)^{-\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s \leqslant C(1+t)^{-\gamma}.$$

Proof. Dividing the integral $\int_0^t (1+t-s)^{-\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s$ by $(1+t)^{-\gamma}$, we derive that

$$\begin{aligned}
\frac{\int_0^t (1+t-s)^{-\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s}{(1+t)^{-\gamma}} &= \int_0^t \left(\frac{1+t-s}{1+t}\right)^{-\gamma} \mathrm{e}^{-\eta s}\,\mathrm{d}s \\
&= \int_0^t \left(1+\frac{s}{1+t-s}\right)^{\gamma} \mathrm{e}^{-\eta s}\,\mathrm{d}s \\
&\leqslant \int_0^t (1+s)^{\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s,
\end{aligned}$$

which ends the proof with the fact that the integral $\int_0^t (1+s)^{\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s$ is bounded by $\int_0^\infty (1+s)^{\gamma}\,\mathrm{e}^{-\eta s}\,\mathrm{d}s$. □

Lemma 4.4 (convolution inequality)**.** *By $*$ denote the convolution. If $u \in L^p(R)$ and $v \in L^q(R)$, then $u * v \in L^r(R)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $1 \leqslant p, q \leqslant \infty$, and*

$$\|u * v\|_{L^r(R)} \leqslant \|u\|_{L^p(R)} \|v\|_{L^q(R)}.$$

In particular, it follows that

$$\|u * v\|_{L^\infty(R)} \leqslant \|u\|_{L^\infty(R)} \|v\|_{L^1(R)};$$

and

$$\|u * v\|_{L^1(R)} \leqslant \|u\|_{L^1(R)} \|v\|_{L^1(R)}.$$

The following Gronwall's inequality is devised for the optimal decay rates. The proof may be conducted by standard approach and details are omitted.

Lemma 4.5. *Assume that the non-negative function $g(t)$ satisfies*

$$g(t) \leqslant M_1(1+t)^{-\mu} \,\mathrm{e}^{-rt} + M_2 \int_0^t \mathrm{e}^{-\delta s} g(s) \,\mathrm{d}s,$$

with non-negative constants M_1, M_2, μ, r and δ. Then it follows that

$$g(t) \leqslant M_1(1+t)^{-\mu} \,\mathrm{e}^{-rt} \exp\left[M_2 \int_0^\infty \mathrm{e}^{-\delta s} \,\mathrm{d}s\right] \leqslant C(1+t)^{-\mu} \,\mathrm{e}^{-rt}.$$

With the above lemmas in hand, we are in a position to prove theorem 4.1.

Proof of theorem 4.1. We plan to use the interpolation to get the L^p-estimate of the solution under the L^1-estimate and L^∞-estimate. We break the proof into three steps. In step 1, we give the L^1-estimate. In step 2, the L^∞-estimate is derived. In step 3, we deduce the L^p-estimate. Next, we perform these three steps respectively.

Step 1 (L^1-estimate)**.** First, taking the Fourier transform to (3.1) with the initial data $(\psi_0(x), \theta_0(x))$ to get an ordinary differential equation system, then solving this ODE directly using standard approach, we arrive at the following solution representation:

$$\hat{\psi}(t,x) = \frac{\mathrm{e}^{-(1-\alpha+\alpha\xi^2)t}}{2\mathrm{i}\sqrt{\nu}} [\mathrm{e}^{\sqrt{\nu}\xi t}(\hat{\theta}_0 + \mathrm{i}\sqrt{\nu}\hat{\psi}_0) - \mathrm{e}^{-\sqrt{\nu}\xi t}(\hat{\theta}_0 - \mathrm{i}\sqrt{\nu}\hat{\psi}_0)] + \frac{1}{2\mathrm{i}\sqrt{\nu}} \int_0^t [\mathrm{e}^{-(1-\alpha+\alpha\xi^2)(t-s)} \{\mathrm{e}^{\sqrt{\nu}\xi(t-s)} - \mathrm{e}^{-\sqrt{\nu}\xi(t-s)}\} \widehat{(\psi\theta)_x}] \,\mathrm{d}s$$

and

$$\hat{\theta}(t,x) = \frac{\mathrm{e}^{-(1-\alpha+\alpha\xi^2)t}}{2} [\mathrm{e}^{\sqrt{\nu}\xi t}(\hat{\theta}_0 + \mathrm{i}\sqrt{\nu}\hat{\psi}_0) - \mathrm{e}^{-\sqrt{\nu}\xi t}(\hat{\theta}_0 - \mathrm{i}\sqrt{\nu}\hat{\psi}_0)] + \frac{1}{2} \int_0^t [\mathrm{e}^{-(1-\alpha+\alpha\xi^2)(t-s)} \{\mathrm{e}^{\sqrt{\nu}\xi(t-s)} + \mathrm{e}^{-\sqrt{\nu}\xi(t-s)}\} \widehat{(\psi\theta)_x}] \,\mathrm{d}s.$$

Taking the inverse Fourier transform to these two identities, we get the solution implicitly reading

$$\psi(t,x) = \frac{1}{2\mathrm{i}\sqrt{\nu}} [K_-(t,x) * (\theta_0 + \mathrm{i}\sqrt{\nu}\psi_0) - K_+(t,x) * (\theta_0 - \mathrm{i}\sqrt{\nu}\psi_0)] + \frac{1}{2\mathrm{i}\sqrt{\nu}} \int_0^t [K_-(t-s,x) - K_+(t-s,x)] * (\psi(s,x)\theta(s,x))_x \,\mathrm{d}s \qquad (4.3)$$

and

$$\psi(t,x) = \frac{1}{2}[K_-(t,x) * (\theta_0 + \mathrm{i}\sqrt{\nu}\psi_0) + K_+(t,x) * (\theta_0 - \mathrm{i}\sqrt{\nu}\psi_0)] + \frac{1}{2}\int_0^t [K_-(t-s,x) + K_+(t-s,x)] * (\psi(s,x)\theta(s,x))_x \,\mathrm{d}s. \tag{4.4}$$

Applying (4.2), making use of convolution inequality and Jensen's inequality, we derive that

$$\begin{aligned}
&\left\| \frac{\partial^m}{\partial x^m}(\psi(t,x), \theta(t,x)) \right\|_{L^1(R,R^2)} \\
&\quad \leqslant C \left\| \frac{\partial^m}{\partial x^m}(K_+(t,x), K_-(t,x)) \right\|_{L^1(R,R^2)} \|(\psi_0(x), \theta_0(x))\|_{L^1(R,R^2)} \\
&\qquad + C\int_0^t \left\| \frac{\partial}{\partial x}(K_+(t-s,x), K_-(t-s,x)) \right\|_{L^1(R,R^2)} \\
&\qquad \times \left\| \frac{\partial^m}{\partial x^m}(\psi(s,x)\theta(s,x)) \right\|_{L^1(R,R^2)} \mathrm{d}s \\
&\quad \leqslant C\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t} + C\int_0^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\,\mathrm{e}^{-2a/s} \sum_{j=0}^m \left\| \frac{\partial^j \theta(s,x)}{\partial x^j} \right\|_{L^1(R)} \mathrm{d}s.
\end{aligned} \tag{4.5}$$

Defining $Y(t) = \sum_{j=0}^k \left\| \frac{\partial^j}{\partial x^j}(\psi(t,x), \theta(t,x)) \right\|_{L^1(R,R^2)}$ and summing (4.5) over $m = 0, \ldots, k$ give us that

$$Y(t) \leqslant C\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t} + C\int_0^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\,\mathrm{e}^{-2a/s}\,Y(s)\,\mathrm{d}s.$$

The application of Gronwall inequality in the above inequality yields that

$$Y(t) \leqslant C\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}.$$

This implies that for any $k \in \mathbb{N}$, it follows that

$$\left\| \frac{\partial^k}{\partial x^k}(\psi(t,x), \theta(t,x)) \right\|_{L^1(R,R^2)} \leqslant C_k\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}. \tag{4.6}$$

Step 2 (L^∞*-estimate*)**.** Notice that given any $U(t,x) \in L^1(R) \cap L^\infty(R)$ for fixed t, we derive from (3.8) by applying the convolution inequality

$$\begin{aligned}
\left\| \frac{\partial^k K_\pm(t-s,x)}{\partial x^k} * U(s,x) \right\|_{L^\infty(R)} &\leqslant \left\| \frac{\partial^k K_\pm(t-s,x)}{\partial x^k} \right\|_{L^\infty(R)} \|U(s,x)\|_{L^1(R)} \\
&\leqslant C(t-s)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)} \|U(s,x)\|_{L^1(R)}.
\end{aligned}$$

On the other hand, from (3.8), it holds that

$$\begin{aligned}
\left\| \frac{\partial^k K_\pm(t-s,x)}{\partial x^k} * U(s,x) \right\|_{L^\infty(R)} &\leqslant \left\| \frac{\partial^k K_\pm(t-s,x)}{\partial x^k} \right\|_{L^1(R)} \|U(s,x)\|_{L^\infty(R)} \\
&\leqslant C\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)} \|U(s,x)\|_{L^\infty(R)}.
\end{aligned}$$

Then the combination of the above two inequalities yields that

$$\begin{aligned}
&\left\| \frac{\partial^k K_\pm(t-s,x)}{\partial x^k} * U(s,x) \right\|_{L^\infty(R)} \\
&\quad \leqslant C(1+t-s)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)} [\|U(s,x)\|_{L^1(R)} + \|U(s,x)\|_{L^\infty(R)}].
\end{aligned}$$

With the solution expressions (4.3) and (4.4), the L^∞-estimates for any order of the derivatives can be estimated as follows:

$$\begin{aligned}
&\left\|\frac{\partial^m}{\partial x^m}(\psi(t,x),\theta(t,x))\right\|_{L^\infty(R,R^2)}\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}(\|(\psi_0(x),\theta_0(x))\|_{L^\infty}+\|(\psi_0(x),\theta_0(x))\|_{L^1})\\
&\qquad+C\int_0^t(1+t-s)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\left(\left\|\frac{\partial^m}{\partial x^m}(\psi(s,x)\theta(s,x))\right\|_{L^\infty(R)}\right.\\
&\qquad\left.+\left\|\frac{\partial^m}{\partial x^m}(\psi(s,x)\theta(s,x))\right\|_{L^1(R)}\right)\mathrm{d}s.
\end{aligned}\tag{4.7}$$

Moreover, we deduce from (4.1) and (4.6) that

$$\begin{aligned}
&\left\|\frac{\partial^m}{\partial x^m}(\psi(s,x)\theta(s,x))\right\|_{L^1(R,R^2)}+\left\|\frac{\partial^m}{\partial x^m}(\psi(s,x)\theta(s,x))\right\|_{L^\infty(R,R^2)}\\
&\quad\leqslant C_m\left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}+\mathrm{e}^{-\frac{a}{2}s}\right]\sum_{j=1}^m\left\|\frac{\partial^j\psi(s,x)}{\partial x^j}\right\|_{L^\infty(R)}.
\end{aligned}\tag{4.8}$$

Applying (4.8) in (4.7) we are led to

$$\begin{aligned}
&\left\|\frac{\partial^m}{\partial x^m}(\psi(t,x),\theta(t,x))\right\|_{L^\infty(R,R^2)}\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}\\
&\qquad+C\int_0^t(1+t-s)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\\
&\qquad\times\left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}+\mathrm{e}^{-\frac{a}{2}s}\right]\sum_{j=1}^m\left\|\frac{\partial^j\psi(s,x)}{\partial x^j}\right\|_{L^\infty(R)}\mathrm{d}s.
\end{aligned}\tag{4.9}$$

If we sum (4.9) with respect to m from 1 to k and define

$$\tilde{Y}(t)=\sum_{j=1}^k\left\|\frac{\partial^j}{\partial x^j}(\psi(t,x),\theta(t,x))\right\|_{L^\infty(R,R^2)},$$

we arrive at

$$\begin{aligned}
\tilde{Y}(t)&\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}+C\int_0^t(1+t-s)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\\
&\quad\times\left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}+\mathrm{e}^{-\frac{a}{2}s}\right]\tilde{Y}(s)\,\mathrm{d}s\\
&\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}+C\int_0^t\left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}+\mathrm{e}^{-\frac{a}{2}s}\right]\tilde{Y}(s)\,\mathrm{d}s.
\end{aligned}\tag{4.10}$$

Therefore, the application of Gronwall's inequality to (4.10) gives us the following estimate for any $k\in\mathbb{N}$,

$$\begin{aligned}
&\left\|\frac{\partial^k}{\partial x^k}(\psi(t,x),\theta(t,x))\right\|_{L^\infty(R,R^2)}\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}\exp\left\{C\int_0^t\left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}+\mathrm{e}^{-\frac{a}{2}s}\right]\mathrm{d}s\right\}\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}}\,\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}.
\end{aligned}$$

Step 3 (L^p-estimate). Finally, with the L^1-estimate and L^∞-estimate in hand, for any $p \in [1, \infty]$, we use the interpolation inequality to get the L^p-estimate by

$$\begin{aligned}
&\left\| \frac{\partial^k}{\partial x^k}(\psi(t,x),\theta(t,x)) \right\|_{L^p(R,R^2)} \\
&\quad \leqslant \left\| \frac{\partial^k}{\partial x^k}(\psi(t,x),\theta(t,x)) \right\|_{L^\infty(R,R^2)}^{\frac{p-1}{p}} \left\| \frac{\partial^k}{\partial x^k}(\psi(t,x),\theta(t,x)) \right\|_{L^1(R,R^2)}^{\frac{1}{p}} \\
&\quad \leqslant C\left[(1+t)^{-\frac{1}{2}} \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}\right]^{\frac{p-1}{p}} \left[\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}\right]^{\frac{1}{p}} \\
&\quad \leqslant C(1+t)^{-\frac{1}{2}+\frac{1}{2p}} \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}.
\end{aligned}$$

This completes the proof.

Remark 4.6. For the general system (1.1), we have proved the global existence and established the decay order for the solution subject to the coefficients constriction. For the special system where $\alpha = \beta$ and $\sigma = 1$, we derive the optimal decay order of the solution. However, the optimal decay rates of the solution to the general system is not obtained yet in that the calculation is too complicated to derive the optimal decay order. It is still interesting to develop novel ideas or computational techniques to deal with such a problem.

Acknowledgment

The research was supported by the F S Chia Scholarship. The author thanks Thomas Hillen for helpful discussions.

References

[1] Adams R A 1975 *Sobolev Space* (NewYork: Academic)
[2] Curry J H, Herring J R, Loncaric J and Orszag S A 1984 Order and disorder in two- and three-dimension Bérnard convection *J. Fluid Mech.* **1** 1–38
[3] Duonandikoetxea J 2001 *Fourier Analysis* (Providence, RI: American Mathematical Society)
[4] Evans L C 1990 *Weak Convergence Methods for Nonlinear Partial Differential Equations (CBMS 74)* (Providence, RI: American Mathematical Society)
[5] Evans L C 1997 *Partial Differential Equations* (Providence, RI: American Mathematical Society)
[6] Hsieh D Y 1987 On partial differential equations related to Lorenz system *J. Math. Phys.* **28** 1589–97
[7] Hsiao L and Jian H Y 1997 Global smooth solutions to the spatically periodic Cauchy problem for dissipative nonlinear evolution equations *J. Math. Anal. Appl.* **213** 262–74
[8] Hsieh D Y, Tang S Q, Wang X P and Wu L X 1996 Dissipative nonlinear evolution equations and chaos *Acta Mech. Sin.* **12** 1–14
[9] Jian H Y and Chen D G 1998 On the Cauchy problem for certain system of semilinear parabolic equation *Acta Math. Sin.* **14** 17–34
[10] Levine H A and Sleeman B D 1997 A system of reaction diffusion equations arsing in the theory of reinforced random walks *SIAM. J. Appl. Math.* **57** 683–730
[11] Lorenz E N 1963 Deterministic non-periodic flows *J. Atmos. Sci.* **20** 130–41
[12] Sparrow C 1982 *The Lorenzs Equations: Bifurcations, Chaos, and Strange Attrators* (New York: Springer)
[13] Smoller J 1983 *Shock Waves and Reaction–Diffusion Equations* (New York, Berlin: Springer)
[14] Tang S Q 1995 Dissipative nonlinear evolution equation and chaos *PhD Thesis* The Hong Kong University of Science and Technology
[15] Zeidler E 1995 *Applied Functional Analysis* (New York: Springer)
[16] Tang S Q and Zhao H J 1999 Nonlinear stability for dissipative nonlinear evolution equations with ellipticity *J. Math. Anal. Appl.* **233** 336–58